NORTHERN BEACHES SECONDARY COLLEGE

MANLY SELECTIVE CAMPUS

Year 12

Trial Examination

2020

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Write your name on the front of every booklet.
- In Questions 11 to 14 show relevant mathematical reasoning and/or calculations.
- NESA approved calculators and templates may be used.
- Weighting: 45%

Section I Multiple Choice

- 10 marks
- Attempt all questions.
- Answer Sheet provided
- Allow about 15 minutes for this section

Section II Free Response

- 60 marks
- Start a separate booklet for each question.
- Each question is of equal value.
- All necessary working should be shown in every question.
- Allow about 1 hour and 45 minutes for this section.

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10.

Q1. Each of the students in an athletics team is randomly allocated their own locker from a row of 100 lockers.

What is the smallest number of students in the team that guarantees that two students are allocated consecutive lockers?

A. 26

B. 34

C. 50

D. 51

Q2. Which one of the following describes a Bernoulli random variable?

A. The times achieved by athletes in a 100-metre race

B. The value of a card chosen from a pack of 52 playing cards

C. Choosing a red pen from a container that contains red, black and blue pens

D. The number of emails a doctor receives in a week

Q3. What is the derivative of $\tan^{-1}\left(\dfrac{x}{2}\right)$?

A. $\dfrac{1}{2(4+x^2)}$

B. $\dfrac{1}{4+x^2}$

C. $\dfrac{2}{4+x^2}$

D. $\dfrac{4}{4+x^2}$

Q4. Consider the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j}$.

Which of the following vectors is parallel to $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}$?

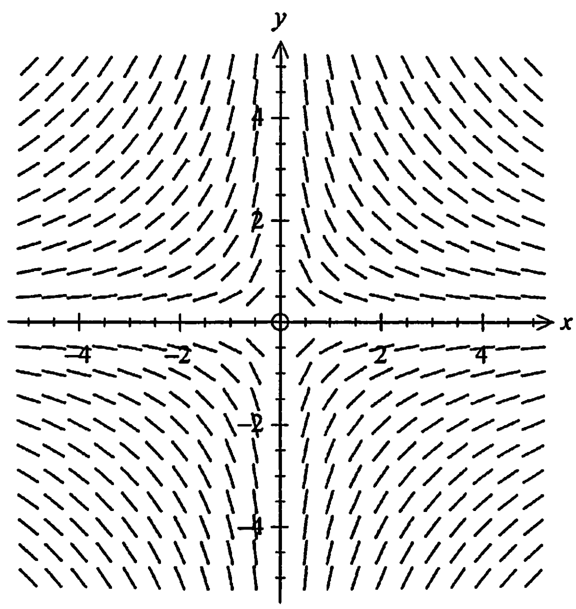
A. $2\underset{\sim}{i} + 8\underset{\sim}{j}$

B. $2\underset{\sim}{i} - 8\underset{\sim}{j}$

C. $2\underset{\sim}{i} - 6\underset{\sim}{j}$

D. $-2\underset{\sim}{i} - 6\underset{\sim}{j}$

Q5. The slope field for a first order differential equation is shown.



What could be the differential equation represented?

A. $\dfrac{dy}{dx} = -\dfrac{x}{y}$

B. $\dfrac{dy}{dx} = \dfrac{x}{y}$

C. $\dfrac{dy}{dx} = -\dfrac{y}{x}$

D. $\dfrac{dy}{dx} = \dfrac{y}{x}$

Q6. Which of the following differential equations can NOT be solved by separation of variables?

A. $\frac{dy}{dx} = x + y$

B. $\frac{dy}{dx} = x$

C. $\frac{dy}{dx} = \frac{x}{y}$

D. $\frac{dy}{dx} = xy$

Q7. What is the value of k such that $\int_0^k \frac{1}{\sqrt{4 - x^2}} dx = \frac{\pi}{3}$?

A. 1

B. $\sqrt{3}$

C. 2

D. $2\sqrt{3}$

Q8. The number of solutions to the equation $(\sin^2 x - 1)(\tan^2 x - 1) = 0$ in the domain $[0, 2\pi]$ is

A. 2

B. 4

C. 6

D. 8

Q9. In how many ways can six people from a group of 15 people be chosen and then arranged in a circle?

A. $\frac{14!}{8!}$

B. $\frac{14!}{8!6}$

C. $\frac{15!}{9!}$

D. $\frac{15!}{9!6}$

Q10. Find $\text{Proj}_{\underset{\sim}{w}} \underset{\sim}{v}$ given $\underset{\sim}{v} = -2\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{w} = 3\underset{\sim}{i} + \underset{\sim}{j}$

A. $-\frac{11}{10}(3i + j)$

B. $-\frac{11}{29}(3i + j)$

C. $-\frac{11}{10}\left(-2\underset{\sim}{i} - 5\underset{\sim}{j}\right)$

D. $-\frac{11}{29}(-2i - 5j)$

End of Multiple Choice

Section II

60 marks

Attempt Questions 11 – 14

Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 **15 marks**

a. On average an archer scores a bullseye once in every three shots at the target. A competitor is allowed eight shots at the target.

i) Find the probability that a bullseye is hit exactly twice. **1**

ii) Find the probability that a bullseye is hit at least twice. **1**

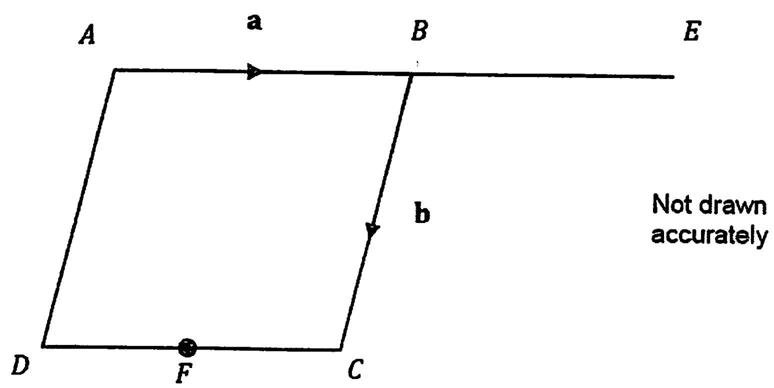
iii) How many shots should the archer use so as to be 90% certain that at least one bullseye is hit? **2**

b. *ABCD* is a rhombus. $\overrightarrow{BE}$ is an extension of $\overrightarrow{AB}$ such that

$|\overrightarrow{AB}| : |\overrightarrow{BE}| = 4 : 3$

$\overrightarrow{AB} = \underset{\sim}{a}$

$\overrightarrow{BC} = \underset{\sim}{b}$



The point *F* is halfway along $\overrightarrow{CD}$.

Find $\overrightarrow{FE}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ **2**

Question 11 continues on next page

Question 11 continued.

c. Find the area enclosed by the curve $y = \tan^{-1} x$, the y-axis and the lines $y = \pm\frac{\pi}{4}$ **3**

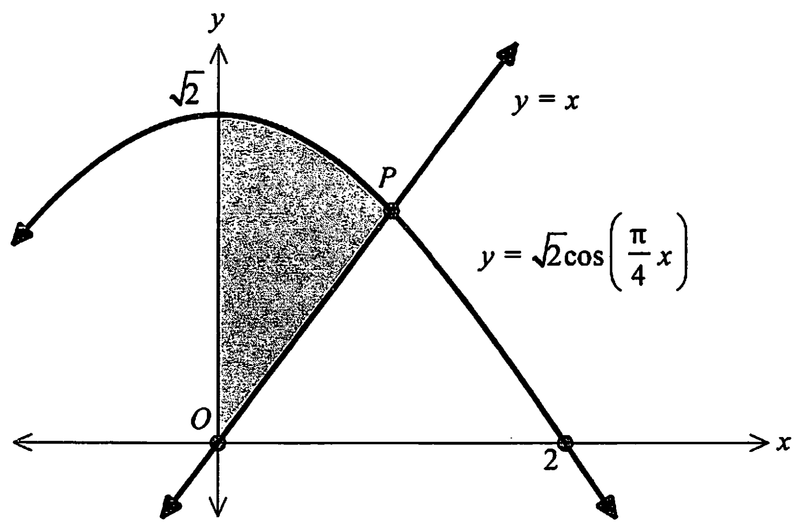
(Express your answer in simplest exact form).

d. Solve the simultaneous equations for x and y leaving your answer in simplest exact form: **2**

$$2\sin^{-1} x + 3\cos^{-1} y = \frac{3\pi}{2}$$

$$2\sin^{-1} x - 3\cos^{-1} y = \frac{\pi}{2}$$

e. The curve $y = \sqrt{2}\cos\left(\frac{\pi}{4}x\right)$ meets the line $y = x$ at the point P, as shown in the diagram.



i) Show that the coordinates of P are (1, 1). **1**

ii) The shaded region is rotated around the x-axis.
Find the volume of the solid of revolution. **3**
(Give your answer in simplest exact form)

End of Question 11

Question 12 **15 marks**

a. Use mathematical induction to prove that $7^{2n-1} + 5$ is divisible by 12 for all integers $n \geq 1$. **3**

b. Solve the differential equation,

$$(x^2 + 5)\frac{dy}{dx} = xy$$

given that $y(2) = 3$ **3**

c. i) Show that $\cos 3x = 4\cos^3 x - 3\cos x$ **2**

ii) Find all solutions to the equation $\cos 3x + 2\cos x = 0$ over the domain $[-2\pi, 2\pi]$ **2**

d. The polynomial $P(x) = x^3 + bx^2 + cx + d$ has zeros 0, 3, and -3.

i) Find b, c and d. **2**

ii) Without using calculus, sketch the graph of $y = P(x)$ **1**

iii) Hence, or otherwise, solve the inequality $\dfrac{x^2 - 9}{x} \geq 0$ **2**

End of Question 12

Question 13 **15 marks**

a. Use the substitution $u = 1 - x$ to evaluate $\int_0^1 x\sqrt{1-x}\,dx$ **3**

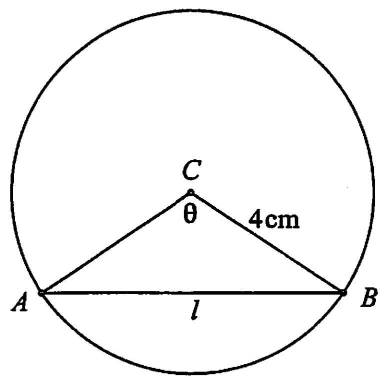
b. i) Express $2\sqrt{3}\sin x - 2\cos x$ in the form $R\sin(x - \alpha)$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ **2**

ii). Find the values of x for which $f(x) = 2\sqrt{3}\sin x - 2\cos x$

attains its maximum value over the domain $[-2\pi, 2\pi]$ **2**

c. In the diagram, C is the centre of a circle of radius 4 cm and $\angle ACB = \theta$ radians.



The length of the chord AB is l cm.

i) Show that the length l of AB is given by $l = 4\sqrt{2 - 2\cos\theta}$ **1**

ii) If θ is increasing at the rate of $\frac{2}{3}$ rad s^{-1}, find the rate of change of the length of AB when $\theta = \frac{\pi}{3}$ radians. Express your answer in simplified exact form. **3**

Question 13 continues on next page.

Question 13 continued.

d. The manufacturers of a chocolate chip cookie are concerned about falling sales. They believe that a fault in their machine means that 15% of the cookies do not have the required number of chocolate chips in them.

A sample of 10 boxes, each containing 20 cookies, is selected.

i) Find the mean and standard deviation for unsatisfactory cookies from the sampling distribution.
(State your answer for the standard deviation to four decimal places.) **2**

ii) What is the probability that more than 20% of the cookies in the sample will be unsatisfactory? (You may assume that the sample of cookies is approximately normally distributed and use a table of z-scores.) **2**

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

End of Question 13.

Question 14 **15 Marks**

a. The population, P, of animals in an environment in which there are scarce resources is increasing in a manner described by the differential equation $\frac{dP}{dt} = P(100 - P)$.

When $t = 0$, $P = 10$.

i) Show that $\frac{1}{100}\left(\frac{1}{P} + \frac{1}{100 - P}\right) = \frac{1}{P(100 - P)}$ **1**

ii) Find an expression for P in terms of t. **3**

b. i) Show that $\frac{d}{dx}\left\{x\tan^{-1}x\right\} = \frac{x}{1 + x^2} + \tan^{-1}x$ **1**

(ii) Hence, show that

$$\int_0^1 \tan^{-1}x \, dx = \frac{\pi}{4} - \log_e(\sqrt{2})$$ **2**

c. A, B and C are points defined by the position vectors $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{c} = \underset{\sim}{i} - 2\underset{\sim}{j}$ respectively.

i) Find $\overrightarrow{AB} \cdot \overrightarrow{CB}$ **1**

ii) Hence, using part i., find the size of $\angle ABC$. **2**

d. Consider the function $f(x) = e^x - \frac{1}{e^x}$

i) Show that $f(x)$ is increasing for all values of x **1**

ii) Find the equation of the inverse function $y = f^{-1}(x)$ **3**

iii) Hence, or otherwise, solve $e^x - \frac{1}{e^x} = 5$.

Give your answer correct to two decimal places. **1**

End of paper

To all students

- Include your ID number on ALL Booklets
- Write in number of booklets used – even if 1 of 1.
- Show full working – especially proof questions. Answers without working frequently are awarded one mark only.
- Do NOT round off unless required.
- Illegible work will NOT be marked

Q1	50 spaces is max possible with one gap therefore 51	D
Q2	Limited to two options Red or Not Red	C
Q3	$\frac{2}{4+x^2}$	C
Q4	$\underset{\sim}{a}=2\underset{\sim}{i}+3\underset{\sim}{j}$, $\underset{\sim}{b}=-3\underset{\sim}{i}+2\underset{\sim}{j}$ and $\underset{\sim}{c}=2\underset{\sim}{i}-\underset{\sim}{j}$. $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}=2\underset{\sim}{i}+3\underset{\sim}{j}-3\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{i}-\underset{\sim}{j}$ $=\underset{\sim}{i}+4\underset{\sim}{j}$ Parallel $k\left(\underset{\sim}{i}+4\underset{\sim}{j}\right)$ eg $2\left(\underset{\sim}{i}+4\underset{\sim}{j}\right)$	A
Q5	$\frac{dy}{dx}=-\frac{y}{x}$ Students – try to match with gradients rather than do the integration.	C
Q6	dy/dx=x+y dy=xdx+ydx It can't be separated	A

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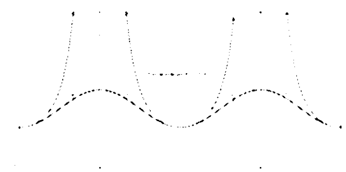
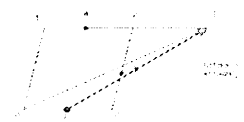
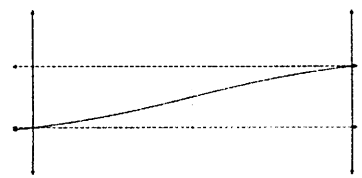
Q7	$\int_0^k \frac{1}{\sqrt{4-x^2}}\,dx=\frac{\pi}{3}$ $\left[\sin^{-1}\frac{x}{2}\right]_0^k=\frac{\pi}{3}$ $\sin^{-1}\frac{k}{2}=\frac{\pi}{3}$ $\frac{k}{2}=\sin\frac{\pi}{3}$ $k=\sqrt{3}$	B
Q8	 Asymptote values (solutions for sin) cannot be valid for tan. 4 solutions	B
Q9	Choose 6 from 15 $= {}^{15}C_6=\frac{15!}{9!\times 6!}$ Arrangements of 6 in a circle = 5! Total arrangements $=\frac{15!}{9!\times 6!}\times 5!$ $=\frac{15!}{9!\times 6}$	D
Q10	Proj$_{\underset{\sim}{w}}\underset{\sim}{v}$ given $\underset{\sim}{v}=-2\underset{\sim}{i}-5\underset{\sim}{j}$ and $\underset{\sim}{w}=3\underset{\sim}{i}+\underset{\sim}{j}$ $\underset{\sim}{v}.\underset{\sim}{w}=-2\times 3-5\times 1=-11$ $\lvert\underset{\sim}{v}\rvert=\sqrt{(-2)^2+(-5)^2}=\sqrt{29}$ $\lvert\underset{\sim}{w}\rvert=\sqrt{3^2+1^2}=\sqrt{10}$ Proj$_{\underset{\sim}{w}}\underset{\sim}{v}=\frac{\underset{\sim}{w}.\underset{\sim}{v}}{\lvert\underset{\sim}{w}\rvert^2}\underset{\sim}{w}$ $=\frac{-11}{10}\left(3\underset{\sim}{i}+\underset{\sim}{j}\right)$	A

Q11		
a-i	$p_{\text{[illegible]}} = \frac{1}{3}$ $(p+q)^8$ Exactly twice $p = \frac{1}{3}\ q = \frac{2}{3}$ $p_2 = {}^8C_2\, p^2 q^6 = \frac{1792}{6561}$	1 mark – correct answer TO ALL STUDENTS – STOP THE ADDICTION TO CALCULATORS AND GIVING YOUR ANSWERS AS ESTIMATES. Exact Fractions are better
a-ii	At least 2 = 1-(0 or 1 hits) $1 - {}^8C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^8 - {}^8C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^7$ $= \frac{5281}{6561}$	1 marks – correct solution
a-iii	no success < 0.1 $\left(\frac{2}{3}\right)^n < 0.1$ $n\log_e\left(\frac{2}{3}\right) < \log_e(0.1)$ $n > \log(0.1) \div \log\left(\frac{2}{3}\right)$ $n > 5.67$ $n = 6$	2 marks – correct solution 1 mark – correct approach for inequality with incorrect final answer.
	Alternatively n shots required. Success on first or success on second or success on third …. $p + qp + qqp + qqqp + \ldots > .9$ $\frac{p(q^n - 1)}{q - 1} > .9$ $\frac{1}{3} \times \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} > .9$ $1 - 0.9 > \left(\frac{2}{3}\right)^n$ $\ln(0.1) > n\ln\left(\frac{2}{3}\right)$ $n > \frac{\ln(0.1)}{\ln\left(\frac{2}{3}\right)}$ $n > 5.67$ $n = 6$	2 marks – correct solution 1 mark – correct approach for inequality with incorrect final answer.

b	 $\Delta - \overrightarrow{DF}$ $\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BF}$ $-\underset{\sim}{b} + \underset{\sim}{a} + \frac{3}{4}\underset{\sim}{a}$ $\overrightarrow{DF} = \frac{1}{2}\underset{\sim}{a}$ $\overrightarrow{DE} - \overrightarrow{DF} = -\underset{\sim}{b} + \frac{7}{4}\underset{\sim}{a} - \frac{2}{4}\underset{\sim}{a}$ $\overrightarrow{FE} = \frac{5}{4}\underset{\sim}{a} - \underset{\sim}{b}$ Or other correct approach.	2 marks – correct solution 1 mark – one correct new vector required for final solution.
c	 Area required is the green section. $y = \tan^{-1}x$ $\tan y = x$ $= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan y \, dy$ Area $= 2\int_0^{\frac{\pi}{4}} \frac{\sin y}{\cos y}\, dy$ $= 2\left[-\ln \cos y\right]_0^{\frac{\pi}{4}}$ $= -2\left(\ln\left(\frac{1}{\sqrt{2}}\right) - \ln 1\right)$ $= -2\ln(\sqrt{2})$ $= -\ln\left(\frac{1}{2}\right)$ $= \ln 2$	3 marks – correct solution 2 marks – transformation and correct integration 1 mark – transformation of integral Too many students did NOT take log laws to the final conclusion.

d	let $\alpha = \sin^{-1}(x)$ $\beta = \cos^{-1} y$ $2\alpha + 3\beta = \frac{3\pi}{2}$ $2\alpha - 3\beta = \frac{\pi}{2}$ $4\alpha = 2\pi \Rightarrow \alpha = \frac{\pi}{2}$ as $-\frac{\pi}{2} \le \sin^{-1}\theta \le \frac{\pi}{2}$ $6\beta = \pi \Rightarrow \beta = \frac{\pi}{6}$ as $0 \le \cos^{-1}\theta \le \pi$ $\frac{\pi}{2} = \sin^{-1}(x)$ $x = 1$ $\frac{\pi}{6} = \cos^{-1} y$ $y = \frac{\sqrt{3}}{2}$	**2 Marks: Correct solution for *x* and *y*** **1 Mark: Correct solution for *x* <u>or</u> *y***
e-i	$P(1,1)$ $y = x \Rightarrow 1 = 1$ ∴ true for $y = x$ $y = \sqrt{2}\cos\left(\frac{\pi}{4}x\right)$ $1 = \sqrt{2}\cos\frac{\pi}{4}$ $1 = \sqrt{2} \times \frac{1}{\sqrt{2}}$ $1 = 1$ true for $y = \sqrt{2}\cos\left(\frac{\pi}{4}x\right)$	**1 Mark: Correct solution**

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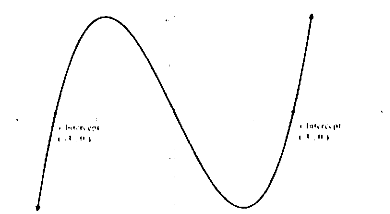
e-ii	$V = \pi\int_a^b [f(x)]^2 - [g(x)]^2 \, dx$ $V = \pi\int_0^1 2\cos^2\left(\frac{\pi}{4}x\right) - x^2 \, dx$ $\cos 2x = 2\cos^2 x - 1$ $\cos 2x + 1 = 2\cos^2 x$ $\therefore 2\cos^2\left(\frac{\pi}{4}x\right) = \cos\left[2\left(\frac{\pi}{4}x\right)\right] + 1$ $V = \pi\int_0^1 \cos\left[\left(\frac{\pi}{2}x\right)\right] + 1 - x^2 \, dx$ $V = \pi\left[\frac{2}{\pi}\sin\frac{\pi x}{2} - x - \frac{x^3}{3}\right]_0^1$ $= \pi\left(\frac{2}{\pi}\sin\frac{\pi}{2} + 1 - \frac{1}{3}\right) - 0$ $= 2 + \frac{2\pi}{3} u^3$	**3 Marks: Correct solution** **2 Marks: Obtains a correct primitive function.** **1 Mark: Obtains a correct integrand expression.**

Question 12

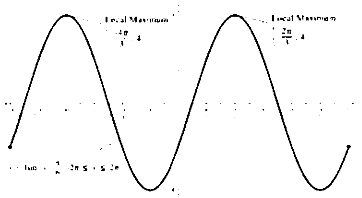
a	$7^{2n-1}+5$ is divisible by 12 $7^{2n-1}+5=12K$ Let $n=1$ LHS $=7^{2-1}+5=12\times1$ True for $n=1$ Assume true for $n=k$ $7^{2k-1}+5=12M$ M element of $\mathbb{Z}$ RTP true for $n=k+1$ $7^{2(k+1)-1}+5=12P$ P element of $\mathbb{Z}$ $LHS=7^{2k-1}+5$ $=7^2\times7^{2k-1}+5$ $=49\{12M-5\}+5$ $=12\times49M+49\times5-5$ $=12\times49M+48\times5$ $=12\{49M+4\times5\}$ $=12P$ where P element of $\mathbb{Z}^+$	**3 Marks: Correct solution** **2 Marks: Correct answer for initial case and correct statements for n=k, n=k+1 and some relevant progress towards showing implication** **1 Mark: Correct answer for initial case and correct statements for n=k & n=k+1**
b	$(x^2+5)\frac{dy}{dx}=xy$ $\frac{1}{y}\,dy=\frac{x}{x^2+5}\,dx$ $\int\frac{1}{y}\,dy=\int\frac{x}{x^2-5}\,dx$ $\ln\|y\|+C_1=\frac{1}{2}\ln\|(x^2+5)\|$ $2\ln\|y\|+C_2=\ln\|(x^2+5)\|$ $x=2$ $y=3$ $2\ln3+C_2=\ln(9)$ $\ln(3^2)+C_2=\ln(9)$ $C_2=0$ $\ln\|y\|=\ln\left(\sqrt{x^2-5}\right)$ $\|y\|=\sqrt{x^2+5}$ $y=\pm\sqrt{x^2+5}$ $y(2)=3\Rightarrow y=\sqrt{x^2+5}$	**3 Marks: Correct solution** **2 Marks: Obtains** $y=\pm\sqrt{x^2+5}$ **OR** **obtains** $y=\sqrt{x^2+5}$ **without justification of positive sign** **OR** **Obtains at least one integral with an absolute value and correctly evaluates a constant** **1 Mark: Obtains at least one correct**

		integral involving absolute value
c-i	$\cos3x=4\cos^3x-3\cos x$ $\cos3x=\cos(2x+x)$ $=\cos2x\cos x-\sin2x\sin x$ $=(2\cos^2x-1)\cos x-2\sin x\cos x\sin x$ $=2\cos^3x-\cos x-2\sin^2x\cos x$ $=2\cos^3x-\cos x-2\times2(1-\cos^2x)\cos x$ $=2\cos^3x-\cos x-2\cos x-2\cos^3x$ $=4\cos^3x-3\cos x$	**2 Marks: Correct solution** **1 Mark: Obtains a correct expression from $\cos3x$ containing $\cos^2x$ and $\sin^2x$, or equivalent expression.**
c-ii	$\cos3x+2\cos x=0$ $4\cos^3x-3\cos x+2\cos x$ $\cos x(4\cos^2x-1)=0$ $\cos x=0$ $x=\pm\frac{\pi}{2},\pm\frac{3\pi}{2}$ $\cos^2x=\frac{1}{4}$ $\cos x=\pm\frac{1}{2}$ $x=\pm\left(\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}\right)$	**2 Marks: Correct solution** **1 Mark: Obtains one set of positive or negative solutions from $\cos x=0$ <u>and</u> $\cos^2x=\frac{1}{4}$** **OR** **Obtains one set of positive <u>and</u> negative solutions from either $\cos x=0$ or $\cos^2x=1/4$**
d-i	$P(x)=x^3+bx^2+cx+d$ $P(x)=x(x-3)(x+3)$ $=x(x^2-9)$ $=x^3-9x$ $b=0$ $c=-9$ $d=0$	**2 Marks: Correct solution** **1 Mark: finds c=-9 OR b=0, d=0**
d-ii		**1 Mark: Correct graph, showing x intercepts**

d-iii	$\frac{x^2-9}{x} > 0$ $\frac{x^2(x^2-9)}{x} > 0 \times x^2$ $x(x^2-9) > 0$ From graph $-3 \leq x < 0$ or $x \geq 3$	**2 Marks: Correct solution** **1 Mark: One correct inequality solution**

Question 13

a	$\int_0^1 x\sqrt{1-x}\,dx$ let $u = 1 - x$ $\therefore \quad x = 1 - u$ $du = -dx$ $x = 0 \Rightarrow u = 1$ $x = 1 \Rightarrow u = 0$ $\int_1^0 (1-u)\sqrt{u}\ \text{-du}$ $= \int_0^1 (1-u)\sqrt{u}\ \text{du}$ $= \int_0^1 u^{\frac{1}{2}} - u^{\frac{3}{2}}\ \text{du}$ $= \left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$ $= \left(\frac{2}{3} - \frac{2}{5}\right) - 0$ $= \left(\frac{10}{15} - \frac{6}{15}\right) = \frac{4}{15}$	3 marks – correct solution 2 marks – correct integration 1 mark - Correct transformed integral with incorrect limits - Incorrect transformed integral with correct limits
b-i	$2\sqrt{3}\sin x - 2\cos x$ $R\sin(x-\alpha) = R\sin x\cos\alpha - R\sin\alpha\cos x$ $R\cos\alpha = 2\sqrt{3}\ R\sin\alpha = 2$ $\tan\alpha = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$ $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ $R = \sqrt{(2\sqrt{3})^2 + 2^2}$ $= \sqrt{16} = 4$ $2\sqrt{3}\sin x - 2\cos x = 4\sin\left(x - \frac{\pi}{6}\right)$	2 marks – both R and α correct by correct method 1 mark – either R or α correct by correct method

b-ii – OPTION 1	Method 1 – graphing  From graph Maximum at $\left(-\frac{4\pi}{3}, 4\right)$ and $\left(\frac{2\pi}{3}, 4\right)$	2 marks – correct solution 1 mark – correct ordinate but failure to check nature. - One max only with nature confirmed
b-ii – OPTION 2	Understanding Trig graph approach For sine curve → max occurs at $\sin\theta = 1$ $\therefore$ $y = 4\sin\left(x - \frac{\pi}{6}\right)$ max at $\sin\left(x - \frac{\pi}{6}\right) = 1,\ 2\pi \le x \le 2\pi$ $x - \frac{\pi}{6} = \frac{\pi}{2}, -\frac{3\pi}{2}$ $x = \frac{2\pi}{3}, -\frac{4\pi}{3}$ $\therefore$ maximum at $\left(\frac{2\pi}{3}, 4\right)$ maximum at $\left(-\frac{4\pi}{3}, 4\right)$	2 marks – correct solution 1 mark – correct ordinate but failure to check nature. - One max only with nature confirmed

b-ii	Calculus approach $y = 4\sin\left(x - \frac{\pi}{6}\right)$ $\frac{dy}{dx} = 4\cos\left(x - \frac{\pi}{6}\right)$ $\frac{d^2y}{dx^2} = -4\sin\left(x - \frac{\pi}{6}\right)$ $4\cos\left(x - \frac{\pi}{6}\right) = 0$ $x - \frac{\pi}{6} = \cos^{-1}(0)$ $x - \frac{\pi}{6} = \pm\frac{\pi}{2}$ or $\pm\frac{3\pi}{2}$ $x = \frac{2\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}, -\frac{4\pi}{3}$ at $x = \frac{2\pi}{3}$ $\frac{d^2y}{dx^2} = -4\left(\sin\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)\right)$ $= -4 \times 1$ $< 0 \therefore$ concave down $\therefore$ maximum at $\left(\frac{2\pi}{3}, 4\right)$ at $x = -\frac{4\pi}{3} \Rightarrow \frac{d^2y}{dx^2} = -4 < 0$ maximum at $\left(-\frac{4\pi}{3}, 4\right)$	2 marks – correct solution 1 mark – correct ordinate but failure to check nature. - One max only with nature confirmed

c-i	$c^2 = a^2 + b^2 - 2ab\cos\theta$ $AB^2 = 4^2 + 4^2 - 2 \times 4 \times 4 \times \cos\theta$ $= 32 - 32\cos\theta$ $AB = \sqrt{16(2 - 2\cos\theta)}$ as $AB > 0$ $= 4\sqrt{2 - 2\cos\theta}$	1 mark – correct demonstration Note lengths > 0
c-ii	Find dl/dt $\frac{dl}{dt} = \frac{d\theta}{dt} \times \frac{dl}{d\theta}$ $\frac{d\theta}{dt} = \frac{2}{3}$ rad s^{-1} $l = 4(2 - 2\cos\theta)^{\frac{1}{2}}$ $\frac{dl}{d\theta} = 4\left[\frac{1}{2} \times 2\sin\theta(2 - 2\cos\theta)^{-\frac{1}{2}}\right]$ $= \frac{4\sin\theta}{\sqrt{2 - 2\cos\theta}}$ $\frac{dl}{dt} = \frac{2}{3} \times \frac{4\sin\theta}{\sqrt{2 - 2\cos\theta}}$ at $\theta = \frac{\pi}{3}$ $\frac{dl}{dt} = \frac{2}{3} \times \frac{4\sin\frac{\pi}{3}}{\sqrt{2 - 2\cos\frac{\pi}{3}}}$ $= \frac{8}{3} \times \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2 - 2 \times \frac{1}{2}}}$ $= \frac{4\sqrt{3}}{3}$ cm s^{-1}	3 marks – correct solution 2 marks - Correct $\frac{dl}{dt}$ 1 mark - Correct $\frac{d\theta}{dt}$ - Correct $\frac{dl}{d\theta}$ **Include units in final answer.**

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d-i	$\mu = p = 0.15$ $\sigma^2 = \frac{p(1-p)}{n}$ $= \frac{0.15 \times 0.85}{200}$ $= 0.0006375$ $\sigma = 0.02524876...$	2 marks – correct solution 1 mark – correct σ^2
d-ii	$z = \frac{x - \mu}{\sigma}$ $= \frac{0.2 - 0.15}{0.02525}$ $= 1.980198..$ $= 1.98$ (2dp) $P(z < 1.98) = 0.9761$ $P(z > 1.98) = 1 - 0.9761$ $= 0.0239$	2 marks – correct solution 1 mark - correct z score

Q14		
a-i	$\frac{1}{100}\left(\frac{1}{P}+\frac{1}{100-P}\right)=\frac{1}{P(100-P)}$ $LHS=\frac{1}{100}\left(\frac{1}{P}+\frac{1}{100-P}\right)$ $=\frac{1}{100}\left[\frac{(100-P)+P}{P(100-P)}\right]$ $=\frac{1}{100}\times\frac{100}{P(100-P)}$ $=\frac{1}{P(100-P)}$	1 mark
a-ii	$\frac{dP}{dt}=P(100-P)$ $\frac{dt}{dP}=\frac{1}{P(100-P)}$ $dt=\frac{1}{P(100-P)}\,dP$ $\int dt=\int\frac{1}{100}\left(\frac{1}{P}+\frac{1}{100-P}\right)dP$ $\int 100dt=\int\frac{1}{P}+\frac{1}{100-P}\,dP$ $100t+C=[\ln\lvert P\rvert-\ln(\lvert 100-P\rvert)]$ $100t-C=\ln\frac{P}{100-P}$ $Ae^{100t}=\frac{P}{100-P}$ when $t=0,\ P=10$ $A=\frac{10}{90}=\frac{1}{9}$ $\frac{1}{9}e^{100t}=\frac{P}{100-P}$ $e^{100t}(100-P)=9P$ $100e^{100t}-Pe^{100t}=9P$ $100e^{t}=P(9+e^{100t})$ $P=\frac{100e^{t}}{9+e^{100t}}$	3 marks 1st mark for correct integration 2nd mark for correct substitution to find A (or C)

b-i	$\frac{d}{dx}(x\tan^{-1}x)$ $u=x \quad v=\tan^{-1}x$ $\frac{du}{dx}=1 \quad \frac{dv}{dx}=\frac{1}{1+x^2}$ $\frac{dy}{dx}=\frac{udv}{dx}+\frac{vdu}{dx}$ $=x\times\frac{1}{1+x^2}+1\times\tan^{-1}x$ $=\frac{x}{1+x^2}+\tan^{-1}x$	1 mark (must clearly show use of the product rule)
b-ii	$\int_0^1\tan^{-1}x$ $\int\frac{d}{dx}(x\tan^{-1}x)\,dx=\int\frac{x}{1+x^2}+\tan^{-1}x\,dx$ $\therefore$ $\int_0^1\tan^{-1}x\,dx=\left[x.\tan^{-1}x\right]_0^1\ \int_0^1\frac{x}{1+x^2\,dx}$ $-\left[x.\tan^{-1}x\right]_0^1\ \left[\frac{1}{2}\ln\lvert 1+x^2\rvert\right]_0^1$ $=\left(\frac{\pi}{4}-0\right)-\frac{1}{2}\ln 2$ $=\frac{\pi}{4}-\ln 2^{\frac{1}{2}}$ $=\frac{\pi}{4}-\ln\sqrt{2}$	2 marks 1 mark for correctly rearranging the result un (i)

c-i	$\overrightarrow{AB} = \left(2\underset{\sim}{i} + \underset{\sim}{j}\right) - \underset{\sim}{i} + 3\underset{\sim}{j})$ $= \underset{\sim}{i} - 2\underset{\sim}{j}$ $\overrightarrow{CB} = \left(2\underset{\sim}{i} + \underset{\sim}{j}\right) - \left(\underset{\sim}{i} - 2\underset{\sim}{j}\right)$ $= \underset{\sim}{i} + 3\underset{\sim}{j}$ $\lvert\overrightarrow{AB}\rvert = \sqrt{5} \qquad \lvert\overrightarrow{CB}\rvert = \sqrt{10}$ $\overrightarrow{AB} \cdot \overrightarrow{CB} = 1 \times 1 + (-2 \times 3) = -5$	1 mark
c-ii	$\cos\theta = \dfrac{\underset{\sim}{m} \cdot \underset{\sim}{n}}{\lvert\underset{\sim}{m}\rvert\,\lvert\underset{\sim}{n}\rvert}$ $= -\dfrac{5}{\sqrt{5}\sqrt{10}}$ $\cos\theta = -\dfrac{1}{\sqrt{2}}$ $\theta = \dfrac{3\pi}{4}$	2 marks 1 mark for using the angle between two vectors formula 1 mark for correctly finding the angle without using (i)
d-i	$f(x) = e^x - \dfrac{1}{e^x} = e^x - e^{-x}$ $f'(x) = e^x - (-e^{-x})$ $= e^x + e^{-x} > 0$ for all x as $e^y > 0$ for all y	1 mark

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d-ii	$y = f^{-1}x$ $f(x) = e^x - \dfrac{1}{e^x}$ $f^{-1}(x) \Rightarrow$ $x = e^y - \dfrac{1}{e^y}$ $x = \dfrac{e^{2y} - 1}{e^y}$ let $p = e^y$ $0 = p^2 - xp - 1$ $p = \dfrac{x \pm \sqrt{x^2 + 4}}{2}$ as $e^y > 0$ then $p > 0$ $\therefore \quad e^y = \dfrac{x + \sqrt{x^2 + 4}}{2}$ $y = \ln\left(\dfrac{x + \sqrt{x^2 + 4}}{2}\right)$	3 marks 1st mark for interchanging x and y 2nd mark for using the quadratic formula (or equivalent) to make y the subject 3rd mark for correctly choosing the + solution
d-iii	Using inverse for $x = 5$ $y = \ln\left(\dfrac{5 - \sqrt{5^2 + 4}}{2}\right) = 1.64723$ therefore solution for $e^x - \dfrac{1}{e^x} = 5$ $x = 1.65$	1 mark

d-iii alternative

DON'T DO TRIAL AND ERROR

$$5 = e^x - \frac{1}{e^x}$$

$$5 = \frac{e^{2x} - 1}{e^x}$$

$$5\,e^x = e^{2x} - 1$$

let $p = e^x$

$$0 = p^2 - 5p - 1$$

$$p = \frac{5 \pm \sqrt{5^2 + 4}}{2}$$

$$e^x = \frac{5 \pm \sqrt{29}}{2}$$

but $e^x > 0$

$$e^x = \frac{5 + \sqrt{29}}{2}$$

$$x = \ln\left(\frac{5 + \sqrt{29}}{2}\right) = 1.6472$$